

# A generalized Lieb-Liniger model

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## Abstract

In 1963, Lieb and Liniger solved exactly a one dimensional model of bosons interacting by a repulsive  $\delta$ -potential and calculated the ground state in the thermodynamic limit. In the present work, we extend this model to a potential of three  $\delta$ -functions, one of them is repulsive and the other two are attractive, modeling some aspects of the interaction between atoms, and present an approximate solution for a dilute gas. In this limit, for low energy states, the results are found to be reduced to the ones of an effective Lieb Liniger model with an effective  $\delta$ -function of strength  $c_{eff}$  and the regime of stability is identified. This may shed light on some aspects of interacting bosons.

## I. INTRODUCTION

The physics of Bose gases is a fascinating and complicated field of research. Since it involves a many-body problem, analytical results are rare and in some parameter regimes, one can use approximations to describe experimental systems with very good accuracy. For example, for a weakly interacting Bose gas, Mean-Field approximation can be used to reduce the many body Hamiltonian into a one-body non-linear Schrödinger Equation, the Gross-Pitaevskii Equation [1–3]. In the opposite limit, a strongly interacting one dimensional Bose gas can be mapped into a gas of free Fermions (Tonks–Girardeau gas, see, for example, [4–7]). Exact solutions in other regimes are highly desired.

Simple models like the Lieb-Liniger (LL) model, that may not have direct experimental realization, may alert us to unexpected physical phenomena that are overlooked when “reasonable approximations” are made and motivate experiments [6–8]. The model introduced in the present work is of this type.

In their seminal work from 1963 [9], Lieb and Liniger managed to solve exactly a one dimensional model for interacting bosons. They considered the Schrödinger equation for  $N$  particles interacting via a  $\delta$ -function potential

$$\left[ -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + c \sum_{\substack{s=1 \\ j>s}}^N \delta(x_j - x_s) \right] \psi = E\psi \quad (1)$$

where  $x_j$  is the coordinate of the  $j$ -th particle and  $c$  is the amplitude of the  $\delta$  function. Making a Bethe ansatz [19]

$$\psi(x_1, \dots, x_N) \propto \sum_P (-1)^{[P]} \exp \left\{ i \sum_{n=1}^N x_n k_{P_n} \right\} \prod_{j>s} \left[ k_{P_j} - k_{P_s} - \frac{imc}{\hbar^2} \text{sign}(x_j - x_s) \right], \quad (2)$$

where  $k_{P_n}$  are the  $k$  vectors obtained by the permutation  $P$  (where  $[P]$  is its parity) of the set  $k_1, \dots, k_N$ . Lieb and Liniger wrote Bethe ansatz equations for the  $k$ 's by imposing periodic boundary conditions on a ring of length  $L$  [10, 11],

$$\exp \{ i k_j L \} = \prod_{h \neq j}^N \frac{\hbar^2 (k_j - k_h) + imc}{\hbar^2 (k_j - k_h) - imc} = \prod_{h \neq j}^N \frac{1 + \frac{imc}{\hbar^2 (k_j - k_h)}}{1 - \frac{imc}{\hbar^2 (k_j - k_h)}}. \quad (3)$$

These  $N$  coupled equations are solved numerically and the energy

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^N k_j^2 \quad (4)$$

is calculated for the ground state and the excitations [9, 12, 13].

In the present work, we study a simple model which takes into account the range of inter-particle interactions without giving up the mathematical simplicity. It is a generalization of the LL model [9] where in addition to the repulsion there is also attraction. It is defined by the Schrödinger equation for  $N$  interacting particles of mass  $m$ ,

$$\left[ -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + c_0 \sum_{\substack{s=1 \\ j>s}}^N \delta(x_j - x_s) + c_l \sum_{\substack{s=1 \\ j>s}}^N \delta(x_j - x_s - l) + c_l \sum_{\substack{s=1 \\ j>s}}^N \delta(x_j - x_s + l) \right] \psi = E\psi \quad (5)$$

where the inter-particle interaction is modeled as a sum of three  $\delta$ -functions: The central one is repulsive ( $c_0 > 0$ ) while the peripheral ones are attractive ( $c_l < 0$ ). This model is inspired by the Van-der-Waals potential which has repulsive and attractive regimes. By adjusting the parameters  $c_0, c_l, l$  of (5), one can model scattering from many inter-particle potentials [14–16].

In section II, we present Bethe ansatz equations for two bosons interacting via three  $\delta$ -functions interaction potential and in section III an approximation is introduced, that allows to extend the LL Bethe ansatz equations to an arbitrary number of particles. The ground state solution for the approximate equations is found in section IV. Section V specifies the parameters of the regime where the gas is stable. The results and their experimental relevance are discussed in section VI.

## II. BETHE ANSATZ EQUATIONS FOR TWO BOSONS INTERACTING VIA THREE $\delta$ -FUNCTIONS INTERACTION POTENTIAL

We start by writing Bethe ansatz equations for a simple case where there are only two bosons. In this case, the equations are intuitive.

Consider two bosons of mass  $m$  trapped on a ring of length  $L$  and interact according to (5). It is convenient to write the wave function  $\psi$  in terms of center of mass coordinate,  $r_1 = (x_1 + x_2)/2$  and relative motion coordinate,  $r_2 = (x_1 - x_2)/2$ ,

$$\psi(r_1, r_2) = \frac{1}{\sqrt{L}} e^{i\tilde{k}_1 r_1} \phi(r_2) \quad (6)$$

where  $\tilde{k}_1 = 2\pi n/L$  and  $n$  is an integer so that periodic boundary conditions are satisfied. At the center of mass frame of reference,  $\tilde{k}_1 = 0$  and the wavefunction of the relative motion,

$\phi(r_2)$ , satisfies the Schrödinger equation

$$\left[ -\frac{\hbar^2}{4m} \frac{\partial^2}{\partial r_2^2} + \frac{1}{2} c_0 \delta(r_2) + \frac{1}{2} c_l \delta(r_2 - l/2) + \frac{1}{2} c_l \delta(r_2 + l/2) \right] \phi(r_2) = E \phi(r_2) \quad (7)$$

which can be written also as

$$\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r_2^2} + c_0 \delta(r_2) + c_l \delta(r_2 - l/2) + c_l \delta(r_2 + l/2) \right] \phi(r_2) = 2E \phi(r_2). \quad (8)$$

As usual in such cases, the wave function takes a different functional form in each of the four intervals  $[-\frac{L}{4}, -\frac{l}{2}]$ ,  $[-\frac{l}{2}, 0]$ ,  $[0, \frac{l}{2}]$  and  $[\frac{l}{2}, \frac{L}{4}]$ . The result for  $\phi(r_2)$  is

$$\begin{aligned} \phi(r_2) = & C \cos(\tilde{k}_2 r_2) + iC \left\{ \text{sign}(r_2) Q_0 \sin(\tilde{k}_2 r_2) + \right. \\ & \left. Q_l \text{sign}(r_2 - l/2) \sin[\tilde{k}_2(r_2 - l/2)] + Q_l \text{sign}(r_2 + l/2) \sin[\tilde{k}_2(r_2 + l/2)] \right\} \end{aligned} \quad (9)$$

where  $C$  is a normalization constant while  $Q_0$  and  $Q_l$  should be determined. They are easily determined for the  $\delta$ -function interaction since the jump of the derivative at the locations of the  $\delta$ -function satisfies

$$\Delta \phi'(r_2^*) \equiv \left. \frac{d\phi(r_2)}{dr_2} \right|_{r_2^*+0^+} - \left. \frac{d\phi(r_2)}{dr_2} \right|_{r_2^*+0^-} = \frac{2mc^*}{\hbar^2} \phi(r_2^*) \quad (10)$$

where  $r_2^* = 0$ ,  $c^* = c_0$  or  $r_2^* = l/2$ ,  $c^* = c_l$ . This results in two equations for  $Q_0$  and  $Q_l$

$$i\tilde{k}_2 Q_0(\tilde{k}_2) = \frac{mc_0}{\hbar^2} \left\{ 1 + 2iQ_l \sin(\tilde{k}_2 l/2) \right\} \quad (11)$$

and

$$i\tilde{k}_2 Q_l = \frac{mc_l}{\hbar^2} \left\{ \cos(\tilde{k}_2 l/2) + i \left[ Q_0 \sin(\tilde{k}_2 l/2) + Q_l \sin(\tilde{k}_2 l) \right] \right\}, \quad (12)$$

leading to

$$Q_l(\tilde{k}_2) = -\frac{i\tilde{k}_2 \hbar^2 mc_l \left[ \cos(\tilde{k}_2 l/2) + \frac{mc_0}{\tilde{k}_2 \hbar^2} \sin(\tilde{k}_2 l/2) \right]}{\tilde{k}_2^2 \hbar^4 - 2m^2 c_0 c_l \sin^2(\tilde{k}_2 l/2) - \tilde{k}_2 \hbar^2 mc_l \sin(\tilde{k}_2 l)}. \quad (13)$$

$\tilde{k}_2$  should be determined to ensure periodic boundary conditions  $\phi'(r_2) = \phi'(r_2 + L/2)$ . In addition,  $\phi(r_2) = \phi(-r_2)$  so that  $\phi'(r_2) = -\phi'(-r_2)$ . Therefore, in particular,  $\phi'(r_2)|_{r_2=L/4}$  must vanish, leading to

$$e^{i\tilde{k}_2 L/2} = \frac{1 - Q_0(\tilde{k}_2) - 2Q_l(\tilde{k}_2) \cos(\tilde{k}_2 l/2)}{1 + Q_0(\tilde{k}_2) + 2Q_l(\tilde{k}_2) \cos(\tilde{k}_2 l/2)}. \quad (14)$$

Now, we return to coordinates  $x_1, x_2$ . For this purpose, we use the relations:  $k_1 = (\tilde{k}_1 + \tilde{k}_2)/2$ ,  $k_2 = (\tilde{k}_1 - \tilde{k}_2)/2$ ,  $r_1 = (x_1 + x_2)/2$  and  $r_2 = (x_1 - x_2)/2$  resulting in

$$\tilde{k}_1 r_1 + \tilde{k}_2 r_2 = k_1 x_1 + k_2 x_2 \quad (15)$$

and

$$\tilde{k}_1 r_1 - \tilde{k}_2 r_2 = k_2 x_1 + k_1 x_2. \quad (16)$$

The function  $\psi$  of (6) takes the form

$$\begin{aligned} \psi(x_1, x_2) = & C [e^{i(k_1 x_1 + k_2 x_2)} + e^{i(k_2 x_1 + k_1 x_2)}] \\ & + C \text{sign}(x_1 - x_2) Q_0(k_1 - k_2) (e^{i(k_1 x_1 + k_2 x_2)} - e^{i(k_2 x_1 + k_1 x_2)}) \\ & + C Q_l(k_1 - k_2) \text{sign}(x_1 - x_2 - l) [e^{i(k_1 x_1 + k_2 x_2)} e^{-i(k_1 - k_2)l/2} - e^{i(k_2 x_1 + k_1 x_2)} e^{i(k_1 - k_2)l/2}] \\ & + C Q_l(k_1 - k_2) \text{sign}(x_1 - x_2 + l) [e^{i(k_1 x_1 + k_2 x_2)} e^{i(k_1 - k_2)l/2} - e^{i(k_2 x_1 + k_1 x_2)} e^{-i(k_1 - k_2)l/2}]. \end{aligned} \quad (17)$$

The periodic boundary condition  $\psi(x_2 + \frac{L}{2}, x_2) = \psi(x_2 - \frac{L}{2}, x_2)$  results in

$$e^{ik_1 L} = \frac{1 - Q_0(k_1 - k_2) - 2Q_l(k_1 - k_2) \cos((k_1 - k_2)l/2)}{1 + Q_0(k_1 - k_2) + 2Q_l(k_1 - k_2) \cos((k_1 - k_2)l/2)} \quad (18)$$

and

$$e^{ik_2 L} = \frac{1 + Q_0(k_1 - k_2) + 2Q_l(k_1 - k_2) \cos((k_1 - k_2)l/2)}{1 - Q_0(k_1 - k_2) - 2Q_l(k_1 - k_2) \cos((k_1 - k_2)l/2)} \quad (19)$$

which are identical to (14) (under the assumption  $\tilde{k}_1 = 0$ , namely, in the center of mass frame of reference). In the derivation we used the fact that  $x_1 - x_2 \rightarrow x_1 - x_2 + L$  involves rotation around the circle and consequently all the signs are changed.

### III. APPROXIMATE BETHE ANSATZ EQUATIONS FOR AN ARBITRARY NUMBER OF BOSONS

The two particle solution cannot be simply generalized to an arbitrary number of particles since for small interparticle distances,

$$|x_j - x_s| < l, \quad (20)$$

the sign function in equation corresponding to (17) varies substantially. For small  $l$ , the effect of the regime (20) may be negligible as demonstrated in what follows. This is reasonable

for a dilute gas where  $l \ll L/N$ . In such a situation, the LL solution is valid with the replacement  $\frac{imc}{\hbar^2(k_j - k_s)} = Q_0 + 2Q_l \cos((k_j - k_s)l/2)$ , leading to

$$\psi(x_1, \dots, x_N) = C \sum_P \left[ \exp \left\{ i \sum_{n=1}^N x_n k_{P_n} \right\} \prod_{j>s} \{1 + (Q_0(k_j - k_s) + 2Q_l(k_j - k_s) \cos((k_j - k_s)l/2)) \text{sign}(x_j - x_s)\} \right] \quad (21)$$

and

$$e^{ik_j L} = \prod_{s \neq j} \frac{1 - Q_0(k_j - k_s) - 2Q_l(k_j - k_s) \cos((k_j - k_s)l/2)}{1 + Q_0(k_j - k_s) + 2Q_l(k_j - k_s) \cos((k_j - k_s)l/2)}. \quad (22)$$

The  $k_j$  are distinct, namely, the wave function vanishes if  $k_j = k_s$  for  $s \neq j$  as was shown in the original work of LL [9].

In the region where inequalities (20) are not satisfied for any of the particle pairs, the sign functions in (9) are all equal. Therefore, in this regime, (21) is a solution with the spectrum (22). There is a Hamiltonian that is different from the original one, for which (21) and (22) are eigenfunctions and eigenvalues even if some of the inequalities (20) are satisfied. It is just defined by the eigenfunctions and eigenvalues. For  $l = 0$ , this Hamiltonian and the original one are identical. If the spectrum and the  $L^2$ -norm of the eigenfunctions are continuous in  $l$ , the relative difference in the spectrum and the wavefunctions (in the  $L^2$ -norm) goes to zero in the limit  $l \rightarrow 0$ . If they are also differentiable as a function of  $l$ , then the relative difference behaves as  $Nl/L$ .

We show that for the low energy states, the 3- $\delta$  function system can be replaced by a system with one  $\delta$ -function of strength  $c_{eff}$ .

We assume

$$(k_j - k_s)l \ll 1 \quad (23)$$

for all wave vectors  $k_j$ . In section V, we show that this limit is relevant for the ground state and low-lying excitations of a dilute gas since  $k_{max} \leq (\text{const}) \frac{N}{L}$  is small. In the leading order in  $k_j l$ ,

$$Q_0 + 2Q_l \cos((k_j - k_h)l/2) \approx -i \frac{m}{\hbar^2 k_2} \left\{ c_0 + 2c_l + \frac{\frac{mc_l l}{\hbar^2} \left[ 2c_0 + 2c_l + \frac{mc_0 c_l}{\hbar^2} + \frac{mc_0^2 l}{2\hbar^2} \right]}{\left[ 1 - \frac{m^2 c_0 c_l l^2}{2\hbar^4} - \frac{mc_l l}{\hbar^2} \right]} \right\} \quad (24)$$

the error is of the order  $Nl/L$ . Comparing (22) with (3), one finds that for small  $k_j l$ , the behavior of the present problem is indeed similar to the one found for one  $\delta$ -function

potential of strength

$$c_{eff} = c_0 + 2c_l + \frac{\frac{mc_l l}{\hbar^2} \left[ 2c_0 + 2c_l + \frac{mc_0 c_l l}{\hbar^2} + \frac{mc_0^2 l}{2\hbar^2} \right]}{\left[ 1 - \frac{m^2 c_0 c_l l^2}{2\hbar^4} - \frac{mc_l l}{\hbar^2} \right]}, \quad (25)$$

in the leading order in  $k_j l$ . Eq. (25) is the main result of the present work, and it enables one to understand the physics of the three  $\delta$ -functions interaction in terms of the one  $\delta$ -function interaction. Of particular interest are situations where  $c_{eff}$  is very different from  $c_0 + 2c_l$  (the total strength of interactions). In order to find such situations, we define the parameters  $r = c_l/c_0$  and  $x = mc_0 l/\hbar^2$  and rewrite (25) as

$$\begin{aligned} \frac{c_{eff}}{c_0 + 2c_l} &= 1 + \frac{rx \left( 2 + 2r + rx + \frac{1}{2}x \right)}{(1 + 2r) \left( 1 - \frac{1}{2}rx^2 - rx \right)} \\ &= \frac{rx + 1 + 2r}{(1 + 2r) \left( 1 - \frac{1}{2}rx^2 - rx \right)} \end{aligned} \quad (26)$$

For weak interactions, ( $x \ll 1$  and  $rx \ll 1$ ),

$$\frac{c_{eff}}{c_0 + 2c_l} \approx 1. \quad (27)$$

However, for very strong interactions ( $x \rightarrow \infty$ ),

$$\frac{c_{eff}}{c_0 + 2c_l} \approx \frac{-2}{(1 + 2r)x} \rightarrow 0^-. \quad (28)$$

This is a surprising result. It is instructive to analyze the behavior of  $c_{eff}/(c_0 + 2c_l)$ , Eq. (26), as a function of  $x$ . We are interested in the regime  $x > 0$  and  $-0.5 < r < 0$ . At  $x = 0$ , the derivative of (26) is negative and therefore the function decreases. At

$$x_0 = -\frac{(1 + 2r)}{r} \quad (29)$$

it turns out that  $c_{eff} = 0$  (even though  $c_0 + 2c_l \neq 0$ ). Higher values of  $x$  result in negative values of  $c_{eff}$ , namely, the effective interaction is attractive (even though  $c_0 + 2c_l > 0$ ). Schematic description of  $c_{eff}/(c_0 + 2c_l)$  is given in Fig. 1.

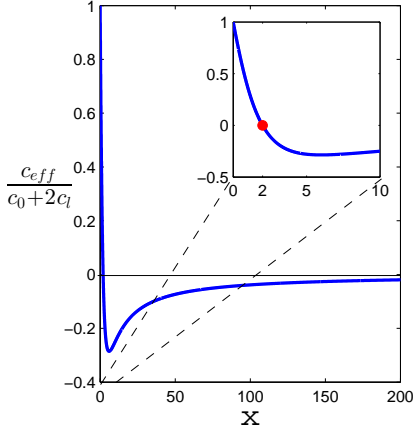


Figure 1: (Color online) Schematic description of  $c_{eff}/(c_0 + 2c_l)$  as a function of  $x$  for  $r = c_l/c_0 = -0.25$ . The inset expands the region where  $c_{eff}$  changes its sign and the red dot is  $(x_0, 0)$ .

The result  $c_{eff} = 0$  at  $x = x_0$  is verified numerically (see Fig. 2) and will be discussed in what follows. In the two particle case it is exact. For a related result see [14, 15].

#### IV. GROUND STATE ENERGY

In the previous section, we derived the approximate Bethe ansatz equations (22) for  $N$  bosons interacting by a three  $\delta$ -function potential (5). The solution for these  $N$  coupled equations,  $(k_1, k_2, \dots, k_N)$ , can be used to calculate the energy of the gas

$$E = \frac{\hbar^2}{2m} \sum_{j=1}^N k_j^2. \quad (30)$$

In the ground state,  $|k_j|$  are minimal (but yet  $k_j$  are different, as in the original work of LL [9]).

Lieb and Liniger [9] managed to calculate the ground state energy in the thermodynamic limit ( $N \rightarrow \infty$ ) by solving only two coupled integral equations (35) and (36) (instead of  $N$  equations of the form (22)). Here, we obtain similar equations by using the logarithmic form of (22),

$$G(k_j) \equiv k_j L + \sum_{s \neq j} \theta(k_j - k_s) = 2\pi \left( n_j - \frac{N+1}{2} \right) \quad (31)$$

where

$$\theta(k) = i \ln \left[ \frac{Q_0(k) + 2Q_l(k) \cos(kl/2) - 1}{Q_0(k) + 2Q_l(k) \cos(kl/2) + 1} \right]. \quad (32)$$



We see that if  $l = 0$ , the ground state corresponds to the choice  $n_j = j$ , ( $j = 1, \dots, N$ ). This is true also for  $l \neq 0$ , as long as  $\theta$  is a monotonic increasing function of  $k$ . To see this, assume  $k_j > k_m$ , then, by monotonicity of  $\theta$ ,  $\theta(k_j - k_s) > \theta(k_m - k_s)$  for all  $s$ , therefore  $G(k_j) > G(k_m)$  and  $G(k_j)$  is monotonic. Since  $\theta$  is an odd function,  $G(k_j = 0) = 0$ . The  $k_j$  for the ground state are the smallest possible in absolute value, hence, we choose  $n_j = j$  for the ground state. Therefore, in the monotonic regime,

$$L(k_{j+1} - k_j) + (k_{j+1} - k_j) \sum_{s \neq j} \theta'(k_j - k_s) = 2\pi \quad (33)$$

where  $\theta'(k) \equiv \partial\theta(k)/\partial k$  and  $k_j$  and  $k_{j+1}$  are adjacent wave numbers. Typically,  $\theta$  is monotonic and (33) is justified at the regime where (23) holds (see Sec. V for more details). The density of states per unit length in  $k$  space, is defined as

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)} \quad (34)$$

and satisfies

$$\int_{-\Lambda}^{\Lambda} dk \rho(k) = \frac{N}{L}. \quad (35)$$

It is used to write (33) in the form

$$\rho(k) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \rho(q) \theta'(k - q) = \frac{1}{2\pi}. \quad (36)$$

Here,  $\Lambda$  is the Fermi momentum (this should not be confused with fermionic systems!). The ground state energy (30) is

$$E_0 = \frac{\hbar^2 L}{2m} \int_{-\Lambda}^{\Lambda} dk \rho(k) k^2. \quad (37)$$

In order to solve Eqs.(35) and (36), we change into dimensionless variables:

$$z = \frac{k}{\Lambda}, \quad \alpha_0 = \frac{c_0 m}{\Lambda \hbar^2}, \quad \alpha_l = \frac{c_l m}{\Lambda \hbar^2}, \quad \gamma_0 = \frac{c_0 m L}{\hbar^2 N}, \quad \gamma_l = \frac{c_l m L}{\hbar^2 N}, \quad d = \Lambda L. \quad (38)$$

In these variables,  $\eta(z) = \theta(\Lambda z)$ , the density of states is  $g(z) = \rho(\Lambda z)$ , and Eqs. (35), (36) and (37) are, respectively [9],

$$\gamma_0 \int_{-1}^1 dz g(z) = \alpha_0, \quad (39)$$

$$g(z) - \frac{1}{2\pi} \int_{-1}^1 dy \eta'(y - z) g(y) = \frac{1}{2\pi} \quad (40)$$

and

$$e \equiv \frac{2m E_0 L^2}{\hbar^2 N^3} = \frac{\gamma_0^3}{\alpha_0^3} \int_{-1}^1 dy g(y) y^2. \quad (41)$$

How does one solve Eqs. (39) and (40)? First, it is necessary to choose values for  $\alpha_0, \alpha_l$  and  $d$ . These values are related to the parameters of the Hamiltonian via the Fermi momentum  $\Lambda$  which is unknown at this stage. One should only keep in mind that  $\alpha_l/\alpha_0 = c_l/c_0$  and therefore the ratio  $\alpha_l/\alpha_0$  does reflect the ratio between attraction and repulsion in the Hamiltonian. The integral equation (40) (with parameters  $\alpha_0, \alpha_l$  and  $d$ ) can be solved numerically. The solution,  $g(z)$ , should be substituted in (39) in order to find  $\gamma_0$ . By repeating the above scheme for different parameters, it is possible to plot the dimensionless energy  $e$  as a function of the dimensionless interaction strengths  $\gamma_0$  and  $\gamma_l$  and the dimensionless length  $d$ . for small values of  $d$ , the energy  $e$  depends only on the effective strength of interaction, that is (25) in dimensionless units,

$$\gamma_{eff} = \gamma_0 + 2\gamma_l + \frac{\alpha_l d (2\gamma_0 + 2\gamma_l + \alpha_l d \gamma_0 + \frac{1}{2}\alpha_0 d \gamma_0)}{1 - \frac{1}{2}\alpha_0 \alpha_l d^2 - \alpha_l d}. \quad (42)$$

The significance of this effective strength of interaction is demonstrated in figure 2. In this figure, we present the solutions  $e(\alpha_0, \alpha_l, d)$  that were calculated by solving (39), (40) and (41). In Fig. 2(a), the energy is plotted as a function of the total interaction strength  $\gamma_0 + 2\gamma_l$  and different choices of  $d$  are represented by different colors. It is clear that the effect of  $d$  is not negligible. Even at the regime  $d \ll 1$ , it is evident that the value of  $l$  has a strong effect on the ground state energy. Furthermore, even for a given value of  $d$ , the total interaction strength  $\gamma_0 + 2\gamma_l$  (which is proportional to  $c_0 + 2c_l$ ) is not in one to one correspondence with the energy and therefore cannot be used to characterize the gas. Fig 2(b) shows that in the regime  $d \ll 1$ , the energy indeed depends only on  $\gamma_{eff}$  of (42). The results are consistent with (26).

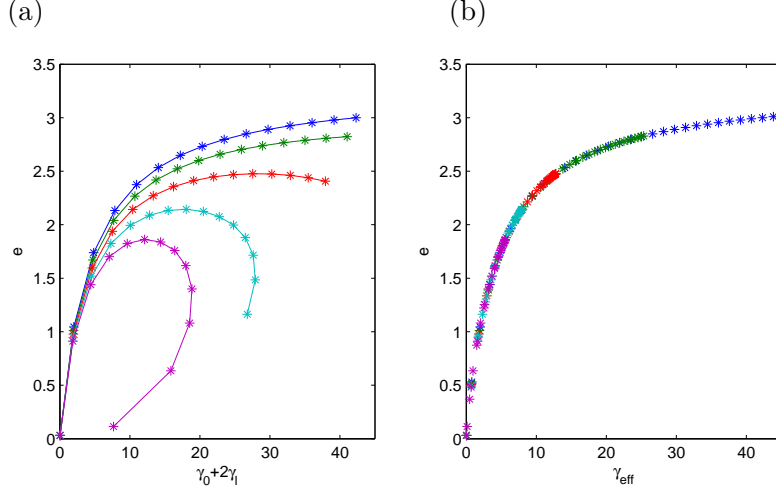


Figure 2: (Color online) The dimensionless energy  $e$  of (41) as a function of dimensionless interaction strengths for  $c_l = -c_0/4$  (namely,  $r = -0.25$ ) and  $0 < \alpha_0 < 30$ . (a)  $e$  as a function of  $\gamma_0 + 2\gamma_l$ . Different lines represent different choices of  $d$  of (38), from top to bottom:  $d = 0$  (blue),  $d = 0.02$  (green),  $d = 0.04$  (red),  $d = 0.06$  (turquoise),  $d = 0.08$  (purple). Points where the effective interaction is attractive were excluded from the figure (these were supposed to appear in the bottom purple curve in the regime  $x > x_0 = 2$ , see Eq. (29)), so that the highest value of  $x$  which does appear in the figure is  $x = 1.94$  and it corresponds to the purple point (7.7, 0.114). (b) The energy  $e$  of (a), plotted as a function of  $\gamma_{\text{eff}}$  (Eq. (42)).

## V. REGIME OF STABILITY AND DEFINITION OF DILUTE GAS

The Bethe ansatz equations (22) and the effective interaction (25), are valid only where (23) is satisfied. Therefore, it is important to identify the regime where  $(k_j - k_s)l \ll 1$ . In the original LL model, the ground state energy and the values of  $k$ 's are maximal for strong interactions,  $c \rightarrow \infty$ , where  $k_n = \frac{2\pi}{L}n$  and  $n$ 's are integers  $n = -\frac{N}{2}, \dots, \frac{N}{2}$ . Then, the maximal absolute value of  $k$  is  $k_{\text{max}} = \frac{\pi N}{L}$  and for all  $j, s$ ,

$$(k_j - k_s)l < \frac{2\pi Nl}{L}. \quad (43)$$

For dilute gas, the inter-particle separation  $L/N$  is much larger than the interaction range  $l$  and (23) is satisfied.

The same argument can be written for the three  $\delta$ -functions interaction potential (5).

If  $\theta$  of (32) is a monotonic increasing function of  $k$ , the ground state is given by  $n_j = j$ , ( $j = 1, \dots, N$ ) and  $k_{max} = k_N$ .

Let us analyze the function  $\theta(k)$  and identify the regime of parameters where it is monotonic. first, note that  $\theta(k)$  is monotonically increasing if and only if  $f(k) \equiv \frac{1}{i} [Q_0 + 2Q_l \cos(kl/2)]$  is monotonically increasing. For  $k \rightarrow 0$ ,  $f(k) = -\frac{m}{\hbar^2 k} c_{eff}$  and therefore it is monotonically increasing as long as  $c_{eff} > 0$ . Hence, if  $c_{eff} > 0$ , there exist some  $k^*$  (which depends on the parameters  $c_0, c_l, l$  and does not depend on  $L$  and  $N$  since  $\theta$  is independent of these variables) such that for all  $k < k^*$ ,  $\theta(k)$  is monotonically increasing. For the ground state, the states with the smallest  $|k_j|$  are occupied, namely,  $n_j = j$  with  $j = 1, \dots, N$ , and

$$G(k_j) < \pi N. \quad (44)$$

$\theta$  is an angle variable and therefore it is bounded (actually, for very small  $k$ ,  $\theta = -\pi$ ). Hence

$$|k_j| < (\text{const}) \frac{N}{L} \quad (45)$$

and can be made arbitrary small. Now, by increasing  $L$  (or decreasing  $N$ ), one may tune the value of  $k_{max}$  such that the conditions

$$k_{max} < k^*$$

and

$$k_{max} l \ll 1$$

are satisfied simultaneously, the regime (23) of dilute gas is achieved and our solution is correct up to a term of order  $Nl/L$ .

For a dilute gas there is a range of parameters where  $c_{eff} > 0$  and the solution is stable. There is also a range of parameters where  $c_{eff} < 0$  and the system is unstable.

## VI. SUMMARY AND DISCUSSION

In this paper, we analyzed a one dimensional dilute Bose gas for an extension of the LL model defined by (5). By dilute gas, we mean that  $l \ll L/N$ , that is, the effective size of a particle  $l$  (for example, the Van-der-Waals radius of an atom) is much smaller than the inter-particle distance. Using this assumption and the Bethe ansatz, we derived the approximate equations for the spectrum (18), (19), (22). In principle, these can be solved

numerically. For low energies in this situation  $|k_j l| \ll 1$  and the model can be approximated by a LL model with one  $\delta$ -function of strength  $c_{eff}$  given by (25) and in dimensionless units by (42). The error of this approximation is of order  $Nl/L$ . This is a good approximation for the dilute gas. The effective interaction  $c_{eff}$  depends on  $c_l$  and  $c_0$  but also on the ratios between the characteristic potential energy scales,  $c_l/l$  and  $c_0/l$ , and the kinetic energy scale,  $\hbar^2/ml^2$ , of a particle trapped in a well of length  $l$ .

Naively one would expect that for small  $k_j$ ,  $c_{eff} \approx c_0 + 2c_l$ . It turns out to be correct for relatively weak interaction energy. For stronger interactions,  $c_{eff}$  becomes very small and even changes its sign (see Fig. 1). Note that this result holds also for the two particle case where it is exact. It is a surprising result, verified numerically in Fig. 2 and its experimental verification should be considered a challenge. The knowledge of  $c_{eff}$  enables to calculate the ground state and the low excited states if the conditions for stability are satisfied. In section IV, the ground state is calculated in the thermodynamic limit for a dilute gas. In particular, it is demonstrated to depend on all parameters via  $c_{eff}$ . We have shown that for a dilute gas there is a regime of parameters where  $c_{eff} > 0$  and therefore the system is stable. For other parameters,  $c_{eff} < 0$  and the dilute gas is unstable. In this regime, the results of [17, 18] regarding dynamics of attractive gas might be realized. If the gas is not dilute, we cannot determine the stability of the system. This theoretical model enables to predict qualitative features of interacting bosons for realistic systems.

The potential (5) can be realized, for example, in optical lattices [6] with tight harmonic trapping along two perpendicular directions ( $E \ll \hbar\omega_\perp$ ) and almost flat potential along the third direction. The inter-particle interactions are in three dimensions and can be modeled by a “delta shell” potential

$$V(r) = \begin{cases} \frac{3c_0\hbar}{4r_{in}^3 m\omega_\perp} & \text{for } r < r_{in} \\ \frac{c_l\hbar}{2r_{out}^2 \varepsilon_{out} m\omega_\perp} & \text{for } r_{out} < r < r_{out} + \varepsilon_{out} \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

Where  $r_{in}, \varepsilon_{out} \rightarrow 0$ . In a previous work [16], we calculated the scattering length  $a$  and the effective range  $r_e$  of the potential (46) (see App.A of [16]), wrote a three dimensional Schrödinger equation and integrated it over two axes to obtain a one dimensional equation of the form (5). This leads to the relations

$$r_{out} = \sqrt{3}l/2, \quad (47)$$

$$a = \frac{1}{4\hbar\omega_{\perp}} (c_0 + 2c_l) \quad (48)$$

and

$$r_e = \frac{2c_l l^2}{a(c_0 + 2c_l)} + \frac{2a}{3}. \quad (49)$$

As seen from (25), for  $l = 0$ ,  $c_{eff}$  is proportional to the scattering length. However, for  $l \neq 0$ ,  $c_{eff}$  cannot be expressed in terms of  $a$  and  $r_e$ . Therefore, it motivates introducing an effective scattering length that dominates the spectrum.

From an experimental point of view, it looks that  $c_{eff}$  is the only quantity that one can measure in order to characterize the inter-particle interactions (because it determines the spectrum). Hence, it makes sense to define an effective scattering length

$$a_{eff} = \frac{c_{eff}}{4\hbar\omega_{\perp}}. \quad (50)$$

This scattering length, which includes corrections originating in the non-vanishing interaction range, is unique for one dimensional bosonic systems.

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